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LETTER TO THE EDITOR

Deformation of the Askey–Wilson algebra with three generators†

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Abstract. A most general algebra with three generators and with ladder and duality properties, GAW(3), is considered. It is shown that the obtained algebra corresponds to (p, q)-deformation of the Askey–Wilson algebra. For the new algebra GAW(3) the ladder representation is constructed and the overlap problem is discussed.

The study of the general algebras with nonlinear commutation relations is an old and unresolved complete problem. Two directions of this research may be detected. The former deals with the raising of the polynomial order of the nonlinearity. A recent expressive example in this direction can be found in [1]. The latter is connected with the input of deformed parameters in the commutators of the considered algebra, i.e. the so-called q- or (p, q)-deformation of algebra (for example, see [2]). Of course, there are articles including both these ideas. In particular, the q-deformed quadratic algebra with three generators (the so-called quadratic Askey–Wilson QAW(3)) was analysed in [3] by a simple, but effective, abstract scheme.

In this letter I would like to find the most general algebra with three generators with important ladder and duality properties [3, 4]. The automorphism of the given algebra as an exchange of two generators between itself, and an appropriate exchange of the structure constants and the deformed parameters, defines the duality property. Creation–annihilation operators as linear combinations of the original one give the probability to build all eigenstates from one state and completely determine the ladder property. Due to the ladder property one has the following relations in the eigenbasis ϕ_n of the generator K_1:

$$K_1\phi_n = \lambda_n\phi_n \qquad K_1\phi_m = \lambda_m\phi_m \qquad \lambda_n \neq \lambda_m$$

$$K_m = [K_1 f_1(K_1) + K_2 f_2(K_1) + K_3 f_3(K_1)]K_n$$

where $f_1(K_1)$, $f_2(K_1)$ and $f_3(K_1)$ are some functions to be established.

It is valid if the new eigenvalue λ_m satisfies the quadratic equation

$$\lambda_n^2 + \lambda_m^2 + A\lambda_m\lambda_n + B(\lambda_m + \lambda_n) + C = 0$$

where A, B and C are some constants and we take into account that $m = n \pm 1$.

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Combining all the above relations with the duality property one obtains the generic solution for the commutation relations of generators K_1, K_2 and K_3 only in the following form:

$$
\begin{aligned}
&d_1K_1K_2 + d_2K_2K_1 = a_0 + a_1K_1 + a_2K_2 + a_3K_3 \\
&d_3K_2K_3 + d_4K_3K_2 \\
&\qquad = b_0 + b_1K_1 + b_2K_2 + b_3K_3 + b_4K_1K_2 + b_5K_2K_1 + b_6K_2^2 + b_7K_2K_1K_2 \\
&d_5K_3K_1 + d_6K_1K_3 \\
&\qquad = c_0 + c_1K_1 + c_2K_2 + c_3K_3 + c_4K_1K_2 + c_5K_2K_1 + c_6K_1^2 + c_7K_1K_2K_1
\end{aligned} \tag{1}
$$

where d_l are deformed parameters and a_l, b_l, c_l are structure constants.

The algebra (1) will be called the general Askey–Wilson algebra with three generators GAW(3) to distinguish the case of QAW(3). One can construct the ladder representation of GAW(3) in the eigenbasis ϕ_n of generator K_1 only from commutation relations:

$$
\begin{aligned}
&K_1 * \phi_n = \lambda_n * \phi_n \qquad K_2 * \phi_n = f_{n+1} * \phi_{n+1} + r_n * \phi_n + f_n * \phi_{n-1} \\
&a_3 * K_3 * \phi_n = [d_1 * \lambda_{n+1} + d_2 * \lambda_n - a_2] * f_{n+1} * \phi_{n+1} \\
&\qquad + [(d_1 + d_2) * \lambda_n * r_n - a_0 - a_1 * \lambda_n - a_2 * r_n] \\
&\qquad * \phi_n + [d_1 * \lambda_{n-1} + d_2 * \lambda_n - a_2] * f_n * \phi_{n-1}.
\end{aligned}
$$

To find the spectrum λ_n and the matrix coefficients f_n and r_n we must demand the following limitations:

$$
\frac{\bar{Y}_1}{Y_1} = \frac{\bar{Y}_2}{Y_2} = \frac{\bar{Y}_3}{Y_3} = \frac{\bar{Y}_4}{Y_4} = \frac{\bar{Y}_5}{Y_5} \tag{2}
$$

where the same notation is introduced:

$$
\begin{aligned}
&Y_1 = d_5d_1 + d_6d_2 - a_3c_7 \qquad Y_2 = d_5d_2 = d_6d_1 \\
&\bar{Y}_1 = d_3d_1 + d_4d_2 - a_3b_7 \qquad \bar{Y}_2 = d_3d_2 = d_4d_1 \\
&Y_3 = -[d_6a_2 + d_1c_3 + a_3c_4] = -[d_5a_2 + d_2c_3 + a_3c_5] \\
&\bar{Y}_3 = -[d_4a_1 + d_1b_3 + a_3b_4] = -[d_3a_1 + d_2b_3 + a_3b_5] \\
&Y_4 = -[a_3c_6 + a_1(d_5 + d_6)] \qquad Y_6 = a_0c_3 - a_3c_0 \\
&\bar{Y}_4 = -[a_3b_6 + a_2(d_3 + d_4)] \qquad \bar{Y}_6 = a_0b_3 - a_3b_0 \\
&Y_5 = a_1c_3 - a_3c_1 - a_0(d_5 + d_6) \qquad Y_7 = a_2c_3 - a_3c_2 \\
&\bar{Y}_5 = a_2b_3 - a_3b_2 - a_0(d_3 + d_4) \qquad \bar{Y}_7 = a_1b_3 - a_3b_1.
\end{aligned}
$$

So, for f_n, r_n and λ_n we have the expressions

$$
\begin{aligned}
&r_n = \frac{Y_4\lambda_n^2 + Y_5\lambda_n + Y_6}{Y_2g_ng_{n+1}} \qquad g_n = \lambda_n - \lambda_{n-1} \\
&\Lambda_n\Lambda_{n-1}f_n^2 = g_{n+1}g_{n-1}r_nr_{n-1} + \bar{Y}_7\lambda_n\lambda_{n-1}/\bar{Y}_2 + \bar{Y}_6(\lambda_n + \lambda_{n-1})/\bar{Y}_2 - Q \\
&Y = \frac{\bar{Y}_3}{Y_1 + 2Y_2} = \frac{\bar{Y}_4}{\bar{Y}_1 + 2\bar{Y}_2} \\
&F = [Y_7 - Y_3Y]/Y_2 \qquad N^2(T - T^{-1})^2 = |F| \\
&\Lambda_n = \lambda_{n+1} - \lambda_{n-1} \qquad \lambda_n = \underline{\lambda}_n - Y \qquad T + T^{-1} + Y_1/Y_2 = 0
\end{aligned}
$$

where $\underline{\lambda}_n = T^{\pm n}$ for $F = 0$, $\underline{\lambda}_n = N[T^n \pm T^{-n}]$ for $F \neq 0$ and Q is the Casimir value.

Now one introduces a new generator K_3'

$$
K_3' = a_0 + a_1K_1 + a_2K_2 + a_3K_3 \tag{3}
$$

and use some limitations (2). Finally, we have the following version of algebra GAW(3):

$$
\begin{aligned}
&d_1K_1K_2 + d_2K_2K_1 = K_3' \\
&d_3K_2K_3' + d_4K_3'K_2 = 2RK_2K_1K_2 + A_1\{K_1, K_2\} + A_2K_2^2 + C_1K_1 + DK_2 + E_1 \qquad (4)\\
&d_3K_3'K_1 + d_4K_1K_3' = 2RK_1K_2K_1 + A_2\{K_1, K_2\} + A_1K_1^2 + C_2K_2 + DK_1 + E_2
\end{aligned}
$$

where $d_2d_3 = d_1d_4$ and $\{.,.\}$ denotes the anticommutator.

Recalling the 'forminvariance' of our algebra (1) to transformations of shift and dilation:

$$K_1 = \beta_1X_1 + \gamma_1 \qquad K_2 = \beta_2X_2 + \gamma_2 \qquad K_3 = \beta_3X_3 + \gamma_3. \tag{5}$$

Applying transformations (3) and (5) we obtain the following commutation relations:

$$
\begin{aligned}
&d_1X_1X_2 + d_2X_2X_1 = X_3' \\
&d_3X_2X_3' + d_4X_3'X_2 = 2RX_2X_1X_2 + \{X_1, X_2\}P_1(\gamma_2)/\beta_2 + X_2^2P_2(\gamma_1)/\beta_1 \\
&\qquad + X_2P_4(\gamma_1, \gamma_2, \gamma_3)/[\beta_1\beta_2] + X_1P_3(\gamma_2)/\beta_2^2 + P_5(\gamma_1, \gamma_2, \gamma_3)/[\beta_1\beta_2^2] \\
&d_3X_3'X_1 + d_4X_1X_3' = 2RX_1X_2X_1 + \{X_1, X_2\}P_2(\gamma_1)/\beta_1 + X_1^2P_1(\gamma_2)/\beta_2 \\
&\qquad + X_1P_4(\gamma_1, \gamma_2, \gamma_3)/[\beta_1\beta_2] + X_2P_6(\gamma_1)/\beta_1^2 + P_7(\gamma_1, \gamma_2, \gamma_3)/[\beta_2\beta_1^2]
\end{aligned}
$$

where $P_l(x_k)$ are polynomials of arguments x_k.

So, one can always reduce any structure constant to an arbitrary quantity except R. In this respect, our algebra GAW(3) definitely differs from previously considered QAW(3).

Let us take $d_1 = p$, $d_2 = -q$ and $d_3 = r$. Finally, equation (4) can be rewritten in the non-symmetrical form

$$
\begin{aligned}
&pK_1K_2 - qK_2K_1 = K_3 \\
&pK_2K_3 - qK_3K_2 = \frac{p}{r}(2RK_2K_1K_2 + A_1\{K_1, K_2\} + A_2K_2^2 + C_1K_1 + DK_2 + E_1) \\
&pK_3K_1 - qK_1K_3 = \frac{p}{r}(2RK_1K_2K_1 + A_2\{K_1, K_2\} + A_1K_1^2 + C_2K_2 + DK_1 + E_2)
\end{aligned}
$$

or in the symmetrical form

$$
\begin{aligned}
&tK_1K_2 - t^{-1}K_2K_1 = K_3 \\
&tK_2K_3 - t^{-1}K_3K_2 = \frac{1}{rq}(2RK_2K_1K_2 + A_1\{K_1, K_2\} + A_2K_2^2 + C_1K_1 + DK_2 + E_1) \\
&tK_3K_1 - t^{-1}K_1K_3 = \frac{1}{rq}(2RK_1K_2K_1 + A_2\{K_1, K_2\} + A_1K_1^2 + C_2K_2 + DK_1 + E_2).
\end{aligned}
$$

We see that the transition from the non-symmetrical to the symmetrical version of GAW(3) can be performed by

$$K_3^{\text{nsym}} = (pq)^{1/2}K_3^{\text{sym}} \qquad \text{and} \qquad t^2 = p/q.$$

In conclusion, for the non-symmetrical version of algebra GAW(3) ($p = r$) we give the explicit expression of the Casimir operator Q commuting with all the generators K_l and briefly discuss the overlap problem:

$$
\begin{aligned}
Q = \left(\frac{p^2+q^2}{2} - R\right)&(-K_1K_2^2K_1 - K_2K_1^2K_2 + A_1K_1K_2K_1 + A_2K_2K_1K_2 + C_2K_2^2 \\
&+ C_1K_1^2 + E_2K_2 + E_1K_1) + (K_1K_2K_1K_2 + K_2K_1K_2K_1 + A_1K_1K_2K_1 \\
&+ A_2K_2K_1K_2 + D\{K_1, K_2\} + E_2K_2 + E_1K_1)pq - A_1A_2\{K_1, K_2\} \\
&- A_1C_2K_2 - A_2C_1K_1.
\end{aligned}
$$

The overlap functions between the dual bases of the generators K_1 and K_2 are the generic Askey–Wilson polynomials $P_n(\mu(x))$, which can be expressed in terms of the basic hypergeometric function ${}_4\Phi_3$ [5]:

$$P_n(\mu(x)) = {}_4\Phi_3 \left\{ \begin{matrix} -n,\ n+1+\alpha+\beta,\ -x,\ x+1+\gamma\delta \\ 1+\alpha,\ 1+\beta,\ 1+\gamma \end{matrix} ;\ T,\ T, \right\}$$

where the argument $\mu(x) = T^{-x} + T^{x+1+\gamma+\delta}$.

There is a complicated correspondence between the polynomial parameters α, β, γ, δ and T and the deformed parameters p, q and structure constants of the GAW(3). The full and detailed examination of this question, and also the representations for the GAW(3), will be considered in the next paper. However, now we represent one simple example for comparison with the results of [3]. In this case one has

$$A_1 = A_2 = 0 \qquad C_1 = C_2 = pq \sinh^2 2W$$

$$\lambda_n = \cosh W(2n+1) \qquad 2\cosh 2W = p/q + q/p - 2R$$

$$f_n^2 = \frac{\mathcal{P}(\lambda_n + \lambda_{n-1})}{g_n^2 \Lambda_n \Lambda_{n-1}} = \frac{\prod_{k=0}^{k=3}(\cosh 2Wp - \cosh 2Wp_k)}{4\sinh^2 2W \sinh^2 2Wn \sinh W(2n-1) \sinh W(2n+1)}$$

where $\cosh 2Wp_k$ are the roots of the characteristic polynomial $\mathcal{P}$ of fourth order of argument $\cosh 2Wp$.

Here, there is, therefore, the following correspondence:

$$T = e^{2W} \qquad \alpha = p_0 + p_2 \qquad \beta = p_0 + p_2$$

$$\gamma = p_0 - p_1 \qquad \delta = p_3 + p_2 \qquad x = s - s_0 = s - (\gamma+\delta)/2$$

(for detail see [3]).

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